A Brief Introduction to the Wigner Distribution

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Abstract

This report is a 5-page quick summary on the most fundamental properties of the Wigner distribution. This document is structured as follows: in the first section the context is given: we introduce the fundamental issues of the time-frequency analysis and the role of joint time-frequencies distributions; then we enumerate the ideal requirements for such distributions. In the second section we specifically define the Wigner distribution and examine which properties it exhibits, which of the above requirements it satisfies and how it compares to the spectrogram.

1 Densities

The fundamental problem of the time-frequency analysis is to discover a good mathematical device able to simultaneously represent a given signal $s(t)$ in terms of its intensity in time and frequency. What we mean by *good* will be clear later.

One of the possible forms in which such a desired mathematical device could be expressed is a *density*. Roughly said, a density (or a distribution) $P(x)$ is a function which expresses how a given quantity distributes in relation to a given variable x per unit of x, such that $P(x)\Delta x$ is the amount that falls in an interval Δx at x, while the total amount is given by

$$\int_{-\infty}^{+\infty} P(x)\,\mathrm{d}x\ ,$$

which is often normalized to unity.

Since most quantities in nature can be expressed as a function of two or more variables, it is wise to introduce two-dimensional (and more-than-two-dimensional) densities. Again, we constrain that a given density $P(x,y)$ is such that the total amount is given by

$$\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} P(x,y)\,\mathrm{d}x\,\mathrm{d}y.$$

Moreover, one can obtain a one-dimensional density for a multi-dimensional one by simply disregarding one or more variables, that is, by integrating out them, as follows

$$P(x) = \int_{-\infty}^{+\infty} P(x,y)\,\mathrm{d}y \qquad \text{and} \qquad P(y) = \int_{-\infty}^{+\infty} P(x,y)\,\mathrm{d}x\ .$$

$P(x)$ and $P(y)$, obtained as above from $P(x,y)$, are said to be the marginal distributions (*marginals*, for short) of $P(x,y)$.

Out of generality, the quantity about which our interest is most concerned is energy, and the variables along which it distributes are time and frequency, thus we drop x and y, and will work explicitly with distributions in the form $P(t,\omega)$ where t represents time and ω frequency.

It is now clear that a candidate for that *good mathematical device* we introduced at the beginning of this section is a time-frequency joint distribution. And it is time to define what we mean for a *good* distribution. A rather comprehensive list of the conditions that a good joint density should satisfy is given below:

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- **positivity**: a density should be positive everywhere

$$\forall t, \omega : P(t, \omega) \geq 0 \ ;$$

  negative densities imply negative energies, go against our sensibility and cannot be interpreted;

- **total energy**: the total energy of the distribution should equal the total energy of the signal:

$$\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} P(t, \omega)\, \mathrm{d}t\, \mathrm{d}\omega = \int_{-\infty}^{+\infty} |s(t)|^2\, \mathrm{d}t = \int_{-\infty}^{+\infty} |S(w)|^2\, \mathrm{d}w \ ;$$

- **marginals**: the marginal distribution obtained integrating over time should equal the energy spectrum, while the one obtained integrating over frequency should equal the istantaneous energy:

$$\int_{-\infty}^{+\infty} P(t, w)\, \mathrm{d}t = |S(\omega)|^2 \quad , \quad \int_{-\infty}^{+\infty} P(t, w)\, \mathrm{d}\omega = |s(t)|^2 \ ;$$

- **time and frequency shift invariance**: shifting a signal by a given amount of time should shift its density in time of the same amount, and the same applies to frequency shifts:

$$s_1(t) = s(t - t_0) \ \Rightarrow \ P_1(t, \omega) = P(t - t_0, \omega) \ ; \ S_1(\omega) = S(\omega - \omega_0) \ \Rightarrow \ P_1(t, \omega) = P(t, \omega - \omega_0)$$

- **finite support**: if the signal is null out of a given time interval, so should do the distribution, and the same if the signal has not frequency components out of a given interval: (weak form):

$$\forall t \notin (t_1, t_2) \ \ s(t) = 0 \ \Rightarrow \ \forall t \notin (t_1, t_2) \ \ P(t, \omega) = 0 \ ,$$

$$\forall \omega \notin (\omega_1, \omega_2) \ \ S(\omega) = 0 \ \Rightarrow \ \forall \omega \notin (\omega_1, \omega_2) \ \ P(t, \omega) = 0 \ .$$

  In strong form: the distribution should be null whenever the signal is null, and the same if the signal has not spectrum components at a give frequency:

$$\exists \bar{t} \mid s(\bar{t}) = 0 \ \Rightarrow \ P(\bar{t}, \omega) = 0 \ ,$$

$$\exists \overline{\omega} \mid S(\overline{\omega}) = 0 \ \Rightarrow \ P(t, \overline{\omega}) = 0 \ .$$

  A distribution respecting the finite support condition in strong form (just expressed) also respects the weak form;

- **global averages**: it should be possible to calculate the average value of any function of time and frequency in the usual way:

$$\langle f(t, \omega) \rangle = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f(t, \omega) P(t, \omega)\, \mathrm{d}\omega\, \mathrm{d}t \ ,$$

  and the result should be meaningful[1];

- **local averages**: the average time for a given frequency and the average frequency at a given time, calculated as conditional averages, should equal the derivative of the signal phase and spectral phase respectively[2]:

$$\langle \omega \rangle_t = \int_{-\infty}^{+\infty} \omega P(t, \omega)\, \mathrm{d}\omega = \phi'(t) \qquad , \qquad \langle t \rangle_\omega = \int_{-\infty}^{+\infty} t P(t, \omega)\, \mathrm{d}t = -\psi'(\omega) \ .$$

[1] Additional considerations on this topic that could be meaningful here are beyond the scope of this document and were not reported. For a complete treatment, see [1], pages 118-120

[2] *Idem*.

2 The Wigner Distribution

Given a signal $s(t)$, the corresponding Wigner distribution is defined by

$$W(t,\omega) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} s^*(t-\frac{1}{2}\tau)\; s(t+\frac{1}{2}\tau)\; \mathrm{e}^{-j\tau\omega}\mathrm{d}\tau$$

or, given the associated spectrum $S(\omega)$ of the signal $s(t)$,

$$W(t,\omega) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} S^*(\omega+\frac{1}{2}\theta)\; S(\omega-\frac{1}{2}\theta)\; \mathrm{e}^{-jt\theta}\mathrm{d}\theta$$

(the two definitions can be easily proven to be equivalent by substituting $s(t)$ with its expression in terms of the spectrum).

The following table summarizes which of the requirements enumerated in the previous section are satisfied by the Wigner distribution, and how it compares to the short-time Fourier transform.

Requirement	Wigner distribution	Short-time Fourier transform
Positivity	no	yes
Total energy	yes	yes
Marginals	yes	no
t,ω shift invariance	yes	yes
Finite support:		
-in weak form	yes	no
-in strong form	no	no
Global averages	no	no
Local averages	yes	no

As far as positivity is concerned, the Wigner distribution fails to meet this requirement, and it can be proven that it must go negative somewhere for every signal (with the only exception of the signals belonging to family of signals illustrated in figure 1; in figure 2 we report the Wigner distribution of that signal).

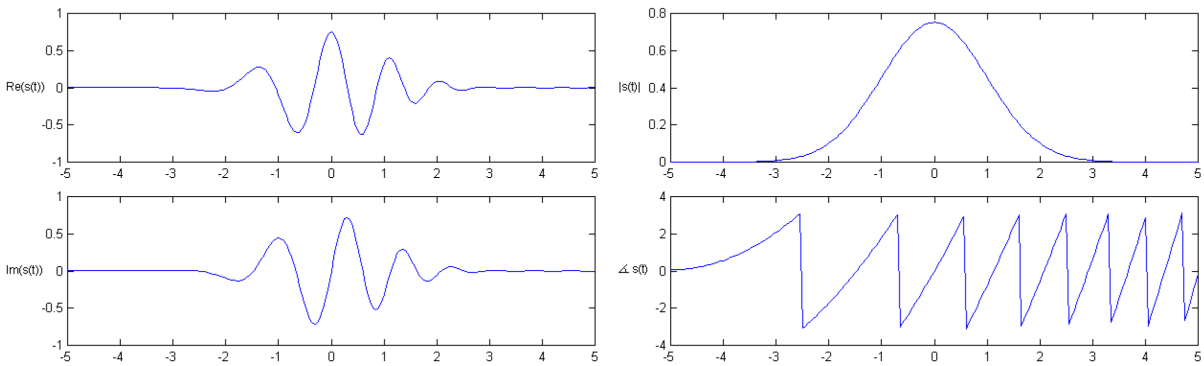

Figure 1: Time representation of an example signal belonging to the only family of signals with positive Wigner distribution, whose equation follows: $s(t) = \sqrt[4]{\alpha/\pi}\ \mathrm{e}^{-\alpha t^2/2+j\beta t^2/2+j\omega_0 t}$; $W(t,\omega) = 1/\pi\ \mathrm{e}^{-\alpha t^2-(\omega-\beta t-\omega_0)^2/\alpha}$. Parameter values for the specific signal plotted are $\alpha = 1, \beta = 1, \omega_0 = 5$.

The Wigner distribution satisfies the marginals requirement, and therefore also the total energy, which is a less strict requirement. The proof is that:

$$\int_{-\infty}^{+\infty} W(t,\omega)\,\mathrm{d}\omega = \frac{1}{2\pi}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} s^*(t-\frac{\tau}{2})s(t+\frac{\tau}{2})\,\mathrm{e}^{-j\tau\omega}\,\mathrm{d}\tau\,\mathrm{d}\omega = \int_{-\infty}^{+\infty} s^*(t-\frac{\tau}{2})s(t+\frac{\tau}{2})\delta(\tau)\,\mathrm{d}\tau = |s(t)|^2 \,,$$

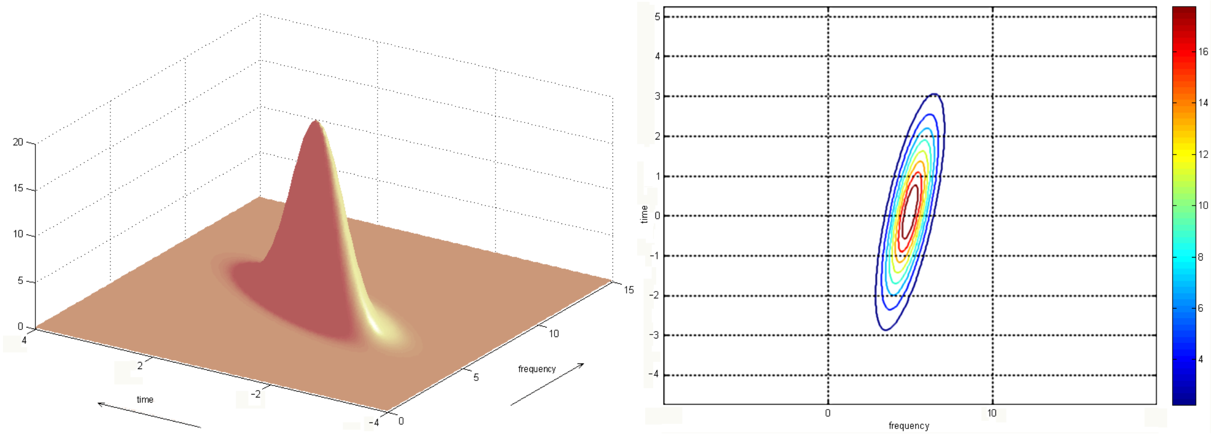

Figure 2: Wigner distribution of the above signal, represented as a surface and as contour curves.

and the same applies to the frequency domain, since the Wigner distribution is identical in the two domains from a formal point of view.

As far as t and ω shift invariance is concerned, the following proof shows that when a signal $s(t)$ (with Wigner distribution $W(t, \omega)$) is shifted in time by t_0 and in frequency by ω_0, thus becoming $s_1(t) = \mathrm{e}^{j\omega_0 t} s(t - t_0)$, its Wigner distribution becomes:

$$W_1(t,\omega) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \mathrm{e}^{-j\omega_0(t-\tau/2)} s(t - t_0 - \frac{\tau}{2})\, \mathrm{e}^{j\omega_0(t+\tau/2)} s(t - t_0 + \frac{\tau}{2})\, \mathrm{e}^{j\omega\tau}\, \mathrm{d}\tau = W(t - t_0, \omega - \omega_0)$$

When it comes to the finite support requirements, a couple of words need to be spent in order to give an intuitive justification of the fact that the weak form is respected but not the strong one. The Wigner distribution inherently implies the intuitive idea of dividing a signal in a right and a left part with respect to time t, and folding the right part over the left (the same holds for the spectrum, again due to the fact that it is formally identical in the two domains). The consequence of such rationale is that the Wigner distribution is unquestionably zero at time t if the signal is always null before t or always null after t (since the either the $s^*(t - \frac{1}{2}\tau)$ or the $s(t + \frac{1}{2}\tau)$ terms of the integrand will be constantly null), thus the weak form is respected. But the Wigner distribution could not be zero at times where the signal is zero, and it could not be zero for frequencies which do not exist in the spectrum (intuitive proof: imagine a signal such that $s(t_0) = 0$ but it is not null somewhere before and after t_0: now fold it at t_0; $W(t_0, \omega)$ will be $\neq 0$). Such a phenomenon is called interference, or cross terms, and it is exemplified in figures 3 and 4. The signal in fig. 3(left) is the sum of two sinusoids whose frequencies $f_1(t)$ and $f_2(t)$ are given in fig. 3(right). The corresponding Wigner distribution is given in fig. 4, in surface and contour representation. The impact of interferences can be easily seen by comparing fig. 4 with fig. 3(right).

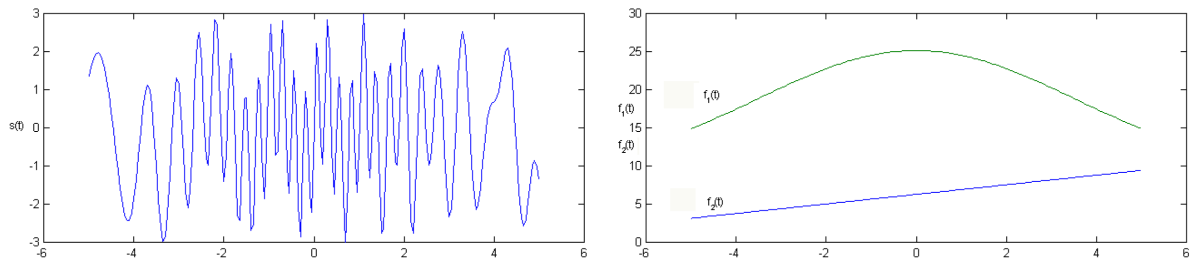

Figure 3: Value and frequency of the following example signal: $s(t) = \sin(f_1(t)) + 2\sin(f_2(t))$, where $f_1 = 2\pi(3 + \sin(t/7\pi - 3\pi/2)$, $f_2(t) = 2\pi(1 + t/10)$

As far as the global averages requirement is concerned, we can say that in general

$$\langle f(t,\omega)\rangle = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f(t,\omega)P(t,\omega)\,\mathrm{d}\omega\,\mathrm{d}t \;\Leftarrow\; f(t,\omega) = f_1(t) + f_2(\omega)\ ,$$

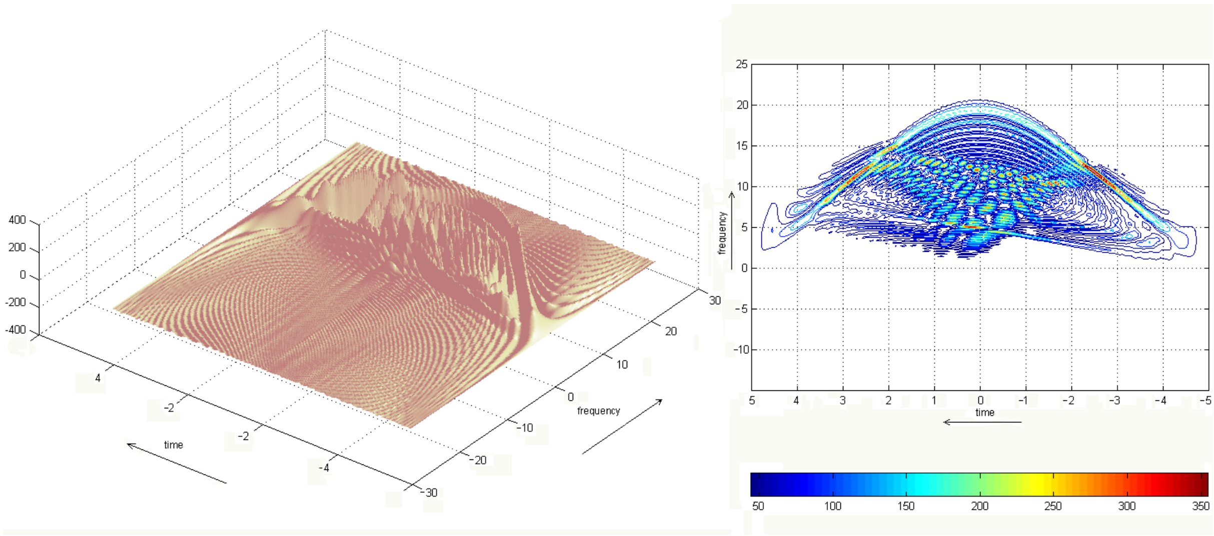

Figure 4: Wigner distribution of the above signal, represented as a surface and as contour curves. Please note the artifacts due to interferences.

that is, the correct average is return only if the function whose average is desired can be split in two components, the first of which is function of t only and the second of ω only. If this hypothesis holds, since the marginals are correct,

$$\langle f(t,\omega)\rangle = \langle f_1(t) + f_2(\omega)\rangle = \int_{-\infty}^{+\infty} f_1(t)|s(t)|^2 \, \mathrm{d}t + \int_{-\infty}^{+\infty} f_2(\omega)|S(\omega)|^2 \, \mathrm{d}\omega \; ,$$

and the correct answer is returned.

When local averages are considered, the first conditional moments of time and frequency are:

$$\langle \omega \rangle_t = \frac{1}{|s(t)|^2} \int_{-\infty}^{+\infty} \omega W(t,\omega) \, \mathrm{d}\omega = \phi'(t)$$

$$\langle t \rangle_\omega = \frac{1}{|S(\omega)|^2} \int_{-\infty}^{+\infty} t W(t,\omega) \, \mathrm{d}t = -\psi'(t) \; ,$$

thus being the instantaneous frequency and the group delay, as desired.

Again, a number of additional properties would be worth mentioning here, but the lack of space makes this impossible.

As a concluding remark, we say that, though not being a proper distribution (since it fails to meet the positivity requirement, whereas the spectrogram is manifestly positive everywhere), nevertheless the Wigner Distribution exhibit some remarkable advantages over the spectrogram: the conditional averages are exactly the instantaneous frequency and the group delay, whereas the spectrogram fails to achieve this result, no matter what window is chosen. Unfortunately, the Wigner distribution of multicomponent signals exhibits the disturbing tendency of generating confusing artifacts, but often the spectrogram, applied to the same signals, fails to provide the resolution required to distinguish the components. Advances found distributions which conserve the advantages of the Wigner distribution and solve most of its drawbacks, but their treatment here is off topic; they can be found in [1].

References

[1] Leon Cohen, *Time-Frequency Analysis*, Prentice Hall PTR

[2] Ananthram Swami, Jerry M. Mendel, Chrysostomos L. Nikias, *Higher-Order Spectral Analysis Toolbox For Use with MATLAB*, The MathWorks, Inc.