Extending formal languages hierarchies to higher dimensions

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Over 30 years ago, it was remarked by M. Minsky ([7]) that pictures could be regarded as sentences in a picture language. This suggested the idea to extend notions and techniques of languages theory to *bi*dimensional (or *multi*dimensional) environments. A generalization of formal languages to two dimensions is possible in different ways, and several formal models to recognize or generate two-dimensional objects have been proposed in the literature. These approaches, initially motivated by problems arising in the framework of pattern recognition and image processing, play also a role in studies concerning cellular automata and other models of parallel computing.

A two-dimensional string is called a *picture* and is defined as a rectangular array of symbols taken from a finite alphabet Σ. A *two-dimensional language* (or *picture language*) is a set of pictures. We first describe two-dimensional languages defined (recognized, generated) by finite-state devices ([1]). We consider definitions that are "natural extensions" from corresponding ones in the string theory, where "natural extension" has the precise meaning that, they reduce to the string case when restricted to pictures of one row (one column) only. The goal is to inherit as many as possible properties from the string theory. A first natural approach is to define picture languages by means of regular expressions. The following (regular) operations are introduced for set of pictures: row and column concatenations, row and column Kleene closures and boolean operations. A regular expression is then a formula expressing how a specific picture language can be obtained from elementary languages by regular operations. Different families of languages can be defined, depending on the choice of operations allowed to be used in the expression. Regarding finite-state machines, several models of them have been designed to recognize picture languages ([3]). As to concerns sequential models, many researches have been devoted to *four-way finite automata (4FA)*, which are the natural generalization of two-way finite automata over strings. Actually these models have a weak computational power, since they operate, more or less explicitely, on some linear coding of a picture and do not capture its true bidimensional features. A more interesting approach considers cellular automata and introduces, in particular, the notion of *two-dimensional on-line tessellation automaton (2-OTA)*. This model can be easily identified to a conventional automaton when restricted to one-row (or one-column) pictures and, moreover, the family of picture languages recognized by a 2-OTA satisfy many important properties. Different systems to generate pictures using grammars have been also explored: however, in the finite-state

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case, this approach is shown to be less powerful than others. Another possible generalization is to describe picture languages by logic formulas. A picture p of size (m, n) over Σ can be viewed as a vertex-labelled grid graph with $m \cdot n$ vertices and then it can be represented as the relational structure $\underline{p} = (dom(p), S_1, S_2, (P_a)_{a \in \Sigma})$ where: $dom(p)$ lists the vertices as coordinate pairs; S_1 and S_2 are the *successor relations* for the two components of points of $dom(p)$, and P_a is set of points in $dom(p)$ that are labelled with a. Properties of pictures can be described by first- or second-order formulas, using first-order variables for points of $dom(p)$, and monadic second-order variables for sets of positions. A further notion of recognizability for sets of pictures is given in terms of *tiling systems*. The underlying idea is to exploit local properties by means of projections. Informally, recognition in a tiling system is defined in terms of a finite set of square pictures of side two, called "tiles", defined over Γ. In a picture over Σ to be recognized, each quadruple of positions that form a square is to be covered by a tile such that a coherent assignment of picture positions to labels in Γ is built up, and such that the projection π from Γ to Σ reestablishes the considered picture. Then the tiles can be viewed as local "automaton transitions", and tiling a given picture means to construct a run of the automaton on it: this is a very "direct" extension from the one-dimensional case where a local language plus a projection correspond exactly to a finite automaton.

The very interesting fact about the definitions above is that some of these different approaches are indeed equivalent ([1, 2]). More precisely: the families of two-dimensional languages recognized by on-line tesselation automata, expressed by formulas of existential monadic second order logic, described by regular expressions of a special type and recognized by finite tiling systems, respectively coincide (i.e. kind of two-dimensional Kleene's and Büchi's Theorems hold). Moreover tiling systems can be equivalently defined, by substituting tiles with dominoes (pictures of sizes $(1, 2)$ and $(2, 1)$): this implies that a picture languages is recognizable if and only if both the languages of all the rows and all the columns, respectively are recognizable as string languages ([4]). All these results show that this notion of recognizability of pictures retains basic attributes from string theory: it can be defined at the same time in terms of machine models, regular expressions, logic formulas and tiling systems. Moreover, some remarkable properties of recognizable string languages (as closure properties, iteration lemma, etc.) can be extended to the two-dimensional case. However, the family of recognizable picture languages is not closed under complementation and the emptiness problem is undecidable: therefore two-dimensional languages are considerably more complicated than string languages, for which non-deterministic is not more powerful than deterministic, and the emptiness problem is decidable.

So far, we have discussed about the finite-state recognizability: in this case case we are able to develop a coherent theory that, unifying different approaches, gives rise to a robust notion of recognizable two-dimensional language. Similarly to one-dimensional case, finite-state devices corresponds to the "ground" level of the theory. In order to built other levels, we first remark that, in spite of the above coincidences, several definitions of finite state recognizability, that are equivalent in the case of dimension $d = 1$, lead to distinct classes when $d \geq 2$. In particular, regular expressions with complementations (even star-free expressions) are able to define not recognizable languages ([6]). In the logic approach, if one substitute the successor relation S with the order relation $\leq$, first-order formulas (resp. existential monadic second-order formulas) define a class of languages incomparable with (resp. containing) that of recognizable ones. Moreover we cannot state a two-dimensional McNaughton-Papert's Theorem: in fact the class of star-free picture languages is stricltly included in the one of first-order definable picture languages ([9]). A general theory, giving the relationships between all these classes is still to built.

As to concern the higher levels of the Chomsky hierarchy, contrary to the finite-state case, there is a lack of connections between all the proposed models ([1]). A natural approach is in terms of grammars ([8]); basically, two models have been proposed called *matrix* and *array grammars*, respectively that define very different families of context-free picture languages. Distinct classes of languages arise also from the approach in terms of automata As further proposal, one can extend the characterizations of finite-states languages that comes from domino systems and define a picture languages to be context-free if and only if both the two languages of all the rows and all the columns, respectively are context-free as string languages. Recently, a new approach in term of grammar was proposed in [5]. Alternatively, one can consider a suitable "two-dimensional Dyck language" and use a "two-dimensional Chomsky–Schützenberger theorem" to define context-free. An interesting problem is to investigate the relationships between all these definitions of context–free picture language.

The purpose of this survey is to introduce the reader to a hierarchy of two-dimensional languages. This hierarchy turns out to be much richer than the corresponding one in one dimension: when moving to higher dimensions, several equivalent definitions generalize in subtle way and give rise to distinct classes. We hope that this hierarchy will allow us to discuss more clearly on complexity issues for images, cellular automata and other systems in two or more dimensions.

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